## **Elementary Concepts in Numerical Analysis**

### Differentiation

- **a)** Prove that the function  $f : \mathbb{R} \to \mathbb{R}$  with f(x) = x is continuous and differentiable.
- **b)** Show mathematically why  $f : \mathbb{R} \to \mathbb{R}$  with f(x) = |x| is not differentiable at  $x_0 = 0$ .
- **c)** Use the chain rule (amongst others) to calculate the derivative of  $2x^3 + x^2 \log 4x$ .
- **d)** Use R to derive the following expressions symbolically:
  - $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4 + 2x^2 + \exp x\right)$
  - $\frac{\mathrm{d}^2}{\mathrm{d}x^2} \left( x^4 + 2x^2 + \exp x \right)$

Calculate the value of the first derivative at point x = 3

- e) Use R to derive the following expressions symbolically:
  - $\frac{\mathrm{d}}{\mathrm{d}x} \left[ x^4 + 2x^2 + \exp x \right]$
  - $\frac{\mathrm{d}^2}{\mathrm{d}x^2} \left[ x^4 + 2x^2 + \exp x \right]$

Calculate the value of the first derivative at point x = 3.

f) Approximate the first derivative at point x = 0.1 via forward, backward and centered differences for

$$f(x) = 3x^4 + 2x$$
 and  $g(x) = \sin \frac{1}{x}$ .

Subsequently, compare the approximated derivatives to their true values. Use  $h = 10^{-6}$ .

g) Calculate the 2nd order central differences at point x = 0.01 for

$$g(x) = \sin\frac{1}{x}.$$

and compare to the true value from a symbolic differentiation. Use  $h = 10^{-6}$ .

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- **h)** Derive a formulate to approximate a Hessian matrix of a function f(x, y) using forward differences. Let h be an arbitrary step size.
- i) Write a user-defined function that can approximate a Hessian matrix of f(x, y) using forward differences. Test your function with  $f(x, y) = 2x + 3xy^2 + y^3 + 1$  at a point  $(x_1, x_2) = (4, 5)$  with a given step size h = 0.0001.

Note: You can also pass function names as arguments; see the following example.

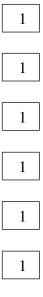
```
f <- function(x) x^2
g <- function(x, func) func(x)+2
g(3, f)
## [1] 11</pre>
```

**j)** Consider the following function  $f(x, y) = 2x + 3xy^2 + y^3 + 1$ . Use the function optimHess in R to approximate the Hessian matrix, i. e. with forward differences at a point  $(x_1, x_2) = (4, 5)$  with a given step size h = 0.0001.

### **Taylor Approximation**

- **k)** Define a Taylor series mathematically.
- I) What is a Maclaurin series?
- **m)** What is the Taylor series for  $e^x$  with  $x_0 = 0$ ?
- **n)** What is the Taylor series for  $\ln 1 x$  with  $x_0 = 0$ ?
- **o)** What is the Taylor series for  $\ln 1 + x$  with  $x_0 = 0$ ?
- **p)** What is the Taylor series for  $\frac{1}{1-x}$  with  $x_0 = 0$ ?

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**q)** What is the 2-nd order Taylor approximation of a function  $f(x, y) = \ln 1 + x + \ln 1 - y$  around  $(x_0, y_0) = (0, 0)$ ?

Hint: a Taylor Series in 2 variables can be written as

$$f(x,y) = f(x_0, y_0) + \left[ (x - x_0) \frac{\partial}{\partial x} + (y - y_0) \frac{\partial}{\partial y} \right] f(x_0, y_0)$$
  
+ 
$$\frac{1}{2!} \left[ (x - x_0) \frac{\partial}{\partial x} + (y - y_0) \frac{\partial}{\partial y} \right]^2 f(x_0, y_0) + \dots$$

**r)** What is the 2-nd order Taylor approximation of a function

$$f(x,y) = \sqrt[5]{x^3 + e^y}$$

around  $(x_0, y_0) = (0, 0)$ ?

- **S)** Calculate the Taylor approximation of  $f(x) = \sin x$  up to degree 4 around  $x_0 = 0$ . Then evaluate and compare it for x = 0.1.
- t) Visualize the function  $f(x) = \log x + 1$  and its Taylor approximation for  $x_0 = 0$ .

#### **Optimality Conditions**

- **u)** Find the stationary points of the function  $f(x, y) = x^3 + 3y y^3 3x$  and analyze their nature using Sylvester's Rule.
- **v)** Consider the function  $f(x, y) = \sin x \cdot \cos x$ . First of all, plot the function nicely to get an impression of its curvature. Then, consider the points

$$\boldsymbol{p}_1 = \begin{bmatrix} rac{\pi}{2} \\ 0 \end{bmatrix}$$
 and  $\boldsymbol{p}_2 = \begin{bmatrix} 0 \\ rac{\pi}{2} \end{bmatrix}$ 

and check their first and second order optimality conditions using R. What type of stationary points do they belong to?

**w)** Prove that the function  $f : \mathbb{R} \to \mathbb{R}$  with  $f(x) = x^2$  is convex.

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